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The free-boundary problem for the die-swell of a viscous fluid

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Abstract. When a viscous fluid is extruded from a capillary or an annular die, the thickness of the fluid jet is in general unequal to the width of the die. This phenomenon is called "die-swell" and is studied in this paper for a die made up of two parallel plates. It is assumed that no slip will occur between the fluid and the plates, and that the pressure in the space into which the fluid is emitted is constant and uniform. The fluid surface is a free streamline. Its shape is calculated with the use of complex-function theory and conformal-mapping techniques. The predicted ratio of swell is found to be in full agreement with known finite-element results.

1. Introduction

For the manufacture of threads and sacks of a thermoplastic material the extrusion process is used. The plastic is melted and extruded from a capillary or an annular die. This fluid is emitted into a gas (e.g. the atmosphere) or another fluid. In that area the product attains its final shape. The intersection of the product will be distinct from the size of the opening of the die or capillary. This difference is known as die-swell; the magnitude of the relative thickness of the product is called the swell-ratio.

In this paper we consider the extrusion of an incompressible Newtonian fluid from a die formed by two parallel plates. This die geometry is an example of a long and narrow strip-like capillary. It also provides a model for an annular die where the flow takes place in the narrow gap between two concentric tubes with large radii. A Newtonian fluid is a linear, homogeneous, isotropic fluid, for which there exists a linear relation between the stress tensor and the strain rate. This type of fluid gives a reasonable indication of the behaviour of visco-elastic fluids, for which the dependence of the stresses on the strain rate is more complicated. The fluid flow is governed by the incompressibility condition, the equations of motion, and the constitutive equations. We restrict ourselves to the isothermal problem, because the influences of the temperature are dominated by the viscous effects. The pressure of the environment, into which the fluid is emitted, is assumed to be constant and uniform. The effects of surface tension are neglected. Further, we assume complete adherence between the fluid and the plates. The velocity field far upstream in the die is the fully developed Poiseuille flow. The surface of the fluid outside the die has an unknown shape. Therefore, the flow problem is a free-boundary problem. Since this boundary must be a free streamline, we have an extra condition to determine the shape of the fluid surface and also the swell-ratio.

For the solution of this free-boundary problem we employ the complex-function theory and the conformal-mapping technique which have successfully been applied to several problems in linear elasticity, see Muskhelishvili [5], England [4]. An application of this theory to viscous fluid flow in injection moulding is given by van Vroonhoven and Kuijpers

[8]. All equations are satisfied by the introduction of two independent analytic functions which are completely determined by the boundary conditions. The free boundary is represented by a conformal mapping which is calculated from the free-streamline condition. In the final section the results are shown and compared to various finite-element simulations listed by Tanner [7].

2. Formulation of the problem

An incompressible Newtonian fluid flows out of a die into an open space where the environmental pressure is constant. The die is a capillary made up of two parallel plates at distance $2a$. The problem is described in the dimensionless Cartesian coordinates x and y, which are related to the usual Cartesian coordinates by $X = ax$ and $Y = ay$. Let B^+ and B^- be the separation points of the fluid from the die. The x-coordinate in these points is chosen to be zero. The plane $y = 0$ corresponds to the plane of symmetry. The y-coordinates of the planes A^+B^+ and A^-B^- are equal to $+1$ and -1 respectively (see Fig. 2.1). The shape of the free boundaries B^+C^+ and B^-C^- is to be determined, especially the swell-ratio h which equals the distance from C^+ (or C^-) to the plane of symmetry divided by a. Far to the left we have the fully developed Poiseuille flow with average velocity V_0. The dimensionless velocity of the fluid is obtained by division by V_0 and is denoted by

$$\mathbf{v} = u(x, y)\mathbf{e}_x + v(x, y)\mathbf{e}_y . \qquad (2.1)$$

The stress tensor in the point (x, y) is denoted by T. The dimensionless stress tensor τ with components t_{xx}, $t_{xy}(=t_{yx})$, t_{yy} is defined by the relation

$$T = \frac{2\eta V_0}{a} \tau ,$$

where η is the viscosity of the fluid.

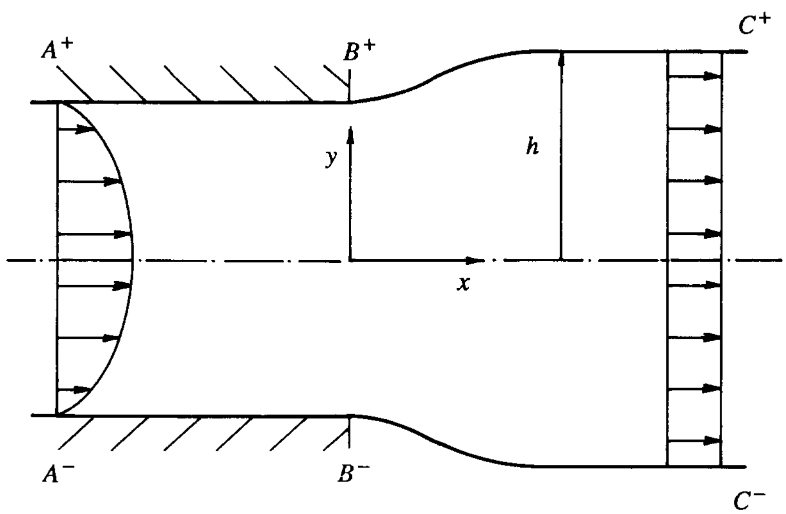

Fig. 2.1. The die geometry.

For an incompressible Newtonian fluid we have the following equations:

(i) the incompressibility condition

$$\operatorname{div} \mathbf{v} = 0\,, \tag{2.2}$$

(ii) the constitutive relation

$$\tau = -pI + d\,, \tag{2.3}$$

where p is the dimensionless hydrostatic pressure and d is the rate of deformation tensor,

$$d = \frac{1}{2}(L + L^T)\,, \quad L = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\,. \tag{2.4}$$

(iii) the conservation of momentum

$$\operatorname{div} \tau^T = 0\,, \tag{2.5}$$

when body forces are absent and the accelerations can be neglected.

As shown in [4, Sec. 2.5], [5, Ch. 5], [8] the equations (2.2) to (2.5) are satisfied by the introduction of the complex variables $z = x + iy$ and $\bar{z} = x - iy$ and of two complex functions $\Omega(z)$ and $\omega(z)$ which are analytic in the domain G_z occupied by the fluid. The general solution of the flow problem is then given by

$$\begin{aligned} w = u + iv &= z\overline{\Omega'(z)} + \overline{\omega'(z)} - \Omega(z)\,, \\ t_{xx} + t_{yy} &= -2[\Omega'(z) + \overline{\Omega'(z)}]\,, \\ t_{xx} - t_{yy} + 2it_{xy} &= 2[z\overline{\Omega''(z)} + \overline{\omega''(z)}]\,. \end{aligned} \tag{2.6}$$

Furthermore, the resulting force over an arc PQ can be expressed as

$$K = -\int_P^Q (t_n + it_s)\,\mathrm{d}z = [z\overline{\Omega'(z)} + \overline{\omega'(z)} + \Omega(z)]_P^Q\,, \tag{2.7}$$

where t_n and t_s are the normal and shear stresses along the arc PQ, see [4, Sec. 2.7], [5, Sec. 33], [8]. The prime $'$ indicates differentiation with respect to the complex argument.

The functions $\Omega(z)$ and $\omega(z)$ are completely determined by the boundary conditions. We need two conditions along every part of the boundary. Along the free surface one extra condition is required, because its shape is unknown.

Complete adherence between the fluid and the planes A^+B^+ and A^-B^- is assumed. This means $\mathbf{v} = \mathbf{0}$ there, and thus

$$u = 0\,, \quad v = 0\,, \qquad y = \pm 1\,, \quad x < 0\,. \tag{2.8}$$

The environmental pressure has a constant (dimensionless) value p_0. Consequently, the normal stress t_n and the shear stress t_s along the free boundaries B^+C^+ and B^-C^- must satisfy

$$t_n = -p_0 , \quad t_s = 0 . \tag{2.9}$$

Substitution of (2.9) into (2.7) yields

$$K = z\overline{\Omega'(z)} + \overline{\omega'(z)} + \Omega(z) = p_0 z + p_1 , \quad z \in B^+C^+ \cup B^-C^- , \tag{2.10}$$

where $p_1 \in \mathbf{C}$ is an integration constant.

The extra condition along the free boundaries B^+C^+ and B^-C^- follows from the fact that it is a streamline, which means that the normal velocity must vanish,

$$(\mathbf{v}, \mathbf{n}) = 0 , \tag{2.11}$$

where $\mathbf{n}$ denotes the outer normal.

For a complete determination of the mathematical problem conditions at infinity ($x \to \pm\infty$) are required. Since the environmental pressure in the open space is constant, we impose that

$$\tau = -p_0 I , \quad (x \to \infty) . \tag{2.12}$$

From this condition it can be derived that there exists a uniform flow at infinity,

$$\mathbf{v} = h^{-1}\mathbf{e}_x , \quad (x \to \infty) , \tag{2.13}$$

where h is the swell-ratio.

In the die the flow will resemble the fully developed Poiseuille flow, which will be denoted by an index 0. Therefore, the limiting value of the velocity must be

$$u \to u_0 = \tfrac{3}{2}(1 - y^2) , \quad v \to v_0 = 0 , \quad (x \to -\infty) . \tag{2.14}$$

Analogous to [8] the Poiseuille flow is subtracted. We write

$$u = u_0 + u_1 , \quad v = v_0 + v_1 , \quad \text{and} \quad w_1 = u_1 + iv_1 . \tag{2.15}$$

The velocities u_0 and v_0 are given by (2.14), while u_1 and v_1 are the new unknown functions. We replace $\Omega(z)$ and $\omega(z)$ by $\Omega_0(z) + \Omega_1(z)$ resp. $\omega_0(z) + \omega_1(z)$ with

$$\Omega_0(z) = -\tfrac{3}{4}(1 + \tfrac{1}{2}z^2) + \tfrac{1}{2}p_2 z , \quad \omega_0(z) = \tfrac{1}{4}z(3 + \tfrac{1}{2}z^2) , \tag{2.16}$$

representing the Poiseuille flow and with $\Omega_1(z)$ and $\omega_1(z)$ the new unknown functions. The constant $p_2 \in \mathbf{R}$ represents a uniform pressure and is still free to be chosen.

The boundary conditions (2.8), (2.10), and (2.14) transform into

$$\begin{aligned} w_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} - \Omega_1(z) = 0 , \quad y = \pm 1, x < 0 , \\ w_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} - \Omega_1(z) \to 0 , \quad (x \to -\infty) , \\ K_1 &= z\overline{\Omega_1'(z)} + \overline{\omega_1'(z)} + \Omega_1(z) \\ &= \tfrac{3}{8}(z^2 + 2z\bar{z} - \bar{z}^2) + (p_0 - p_2)z + p_1 , \quad z \in B^+C^+ \cup B^-C^- . \end{aligned} \tag{2.17}$$

Choosing $p_2 = p_0$ and omitting the irrelevant constant p_1, we have

$$K_1 = \tfrac{3}{8}(z^2 + 2z\bar{z} - \bar{z}^2), \quad z \in \mathrm{B}^+\mathrm{C}^+ \cup \mathrm{B}^-\mathrm{C}^- . \tag{2.18}$$

The equations (2.17) and (2.18) show great resemblance with the formulation of the free-boundary problem in injection moulding as stated in [8]. Therefore, the procedure for the solution of the die-swell problem will be completely analogous. There are only two differences between these free-boundary problems. Firstly, the die-swell geometry stretches out to infinity at two sides. Secondly, the subtracted Poiseuille flow slightly differs because of the use of a moving frame of reference in [8].

3. The conformal mapping

The problem stated above will be treated with conformal-mapping techniques as has been done in [8]. The domain G_z occupied by the fluid is transformed into the interior of the unit circle, $G_\zeta^+ := \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ (see Figs 3.1 and 3.2).

The mapping function is denoted by

$$z = \mathrm{m}(\zeta) . \tag{3.1}$$

Since this transformation is conformal, the function $\mathrm{m}(\zeta)$ is analytic and univalent for $\zeta \in G_\zeta^+$. Further, the mapping function is assumed to be continuous on $\overline{G_\zeta^+}$, except in the points A, $\zeta = -1$, and C, $\zeta = +1$, where logarithmic singularities occur.

As a result from the Riemann mapping theorem the conformal mapping function exists and is uniquely determined by the choice of the points $\mathrm{A}(=\mathrm{A}^+, \mathrm{A}^-)$, B^+ and $\mathrm{C}(=\mathrm{C}^+, \mathrm{C}^-)$ on the unit circle. The following relation for the normal vector $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$ along the boundary of G_z can be derived, see [8],

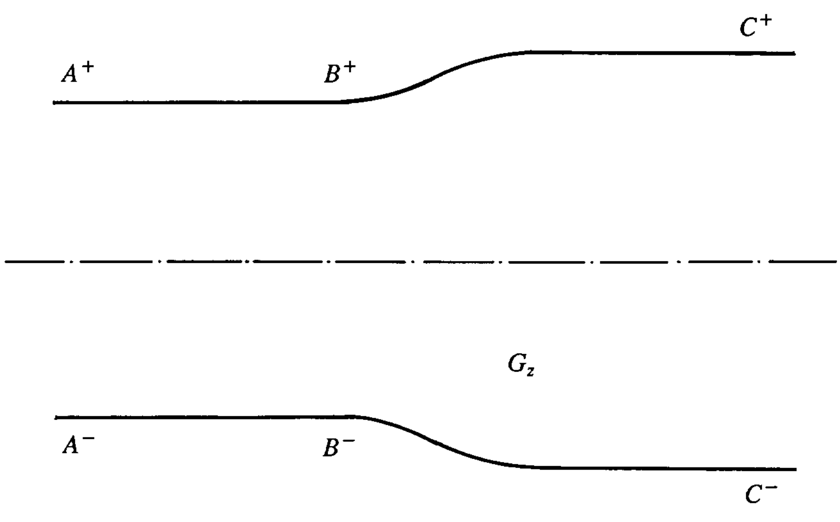

Fig. 3.1. The domain G_z.

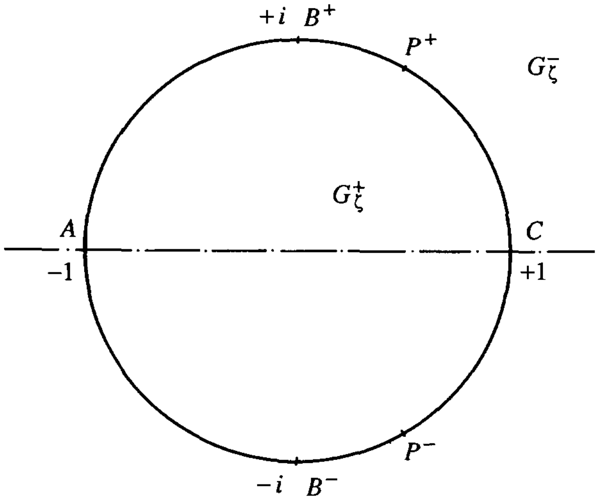

Fig. 3.2. The ζ-plane.

$$\frac{n_x + in_y}{n_x - in_y} = \frac{\xi \mathrm{m}'^+(\xi)}{\overline{\xi \mathrm{m}'^+(\xi)}}, \tag{3.2}$$

where $\mathrm{m}'^+(\xi)$ denotes the limiting value of the derivative $\mathrm{m}'(\zeta)$ for $\zeta \in G_\zeta^+$ tending to a point ξ on the unit circle, $|\xi| = 1$, $\xi \neq \pm 1$.

We follow the conformal-mapping technique and the method of analytic continuation to the exterior of the unit circle $G_\zeta^- := \{\zeta \in \mathbb{C} \mid |\zeta| > 1\}$, as applied to certain problems in the theory of linear elasticity [4, Ch. 5], [5, Ch. 15, 21] and to a free-boundary problem in viscous flow theory [8]. The conformal mapping function $\mathrm{m}(\zeta)$ is approximated by a polynomial $\mathrm{m}_N(\zeta)$ of degree N,

$$\mathrm{m}_N(\zeta) = \sum_{k=0}^{N} \mu_k \zeta^k . \tag{3.3}$$

For reasons of symmetry the coefficients μ_k, $0 \leq k \leq N$, are real but yet unknown.

The points $\zeta = \pm 1$ are mapped onto infinity in the complex z-plane by the exact mapping function $z = \mathrm{m}(\zeta)$, whereas the polynomial $\mathrm{m}_N(\zeta)$ remains finite in these points. From this we conclude that the polynomial $\mathrm{m}_N(\zeta)$ can only produce a good approximation of $\mathrm{m}(\zeta)$ near the separation points B^+ and B^-, while the approximation will not suffice near the points A, $\zeta = -1$, and C, $\zeta = +1$. This assertion can be formalized by the introduction of the points P^+, $\zeta = \mathrm{e}^{i\alpha}$, and P^-, $\zeta = \mathrm{e}^{-i\alpha}$, $0 < \alpha < \frac{1}{2}\pi$, on the unit circle, having the following properties. Firstly, the points P^+ and P^- are mapped onto two points of the free boundaries $\mathrm{B}^+\mathrm{C}^+$ and $\mathrm{B}^-\mathrm{C}^-$ by the exact conformal mapping function $z = \mathrm{m}(\zeta)$. Therefore, we assume that a reliable approximation of the free boundary is given by the image of the arcs $\mathrm{B}^+\mathrm{P}^+$ and $\mathrm{B}^-\mathrm{P}^-$ under the polynomial mapping function $z = \mathrm{m}_N(\zeta)$, i.e. by

$$z = \mathrm{m}_N(\mathrm{e}^{i\theta}) = \sum_{k=0}^{N} \mu_k \,\mathrm{e}^{ik\theta}, \quad \alpha \leq |\theta| \leq \tfrac{1}{2}\pi . \tag{3.4}$$

Secondly, the image of the arc $\mathrm{P}^-\mathrm{C}\mathrm{P}^+$ ($\zeta = \mathrm{e}^{i\theta}$, $-\alpha \leq \theta \leq \alpha$) under the approximate mapping $z = m_N(\zeta)$ is a bounded curve in the complex z-plane (see Fig. 3.3). This curve will

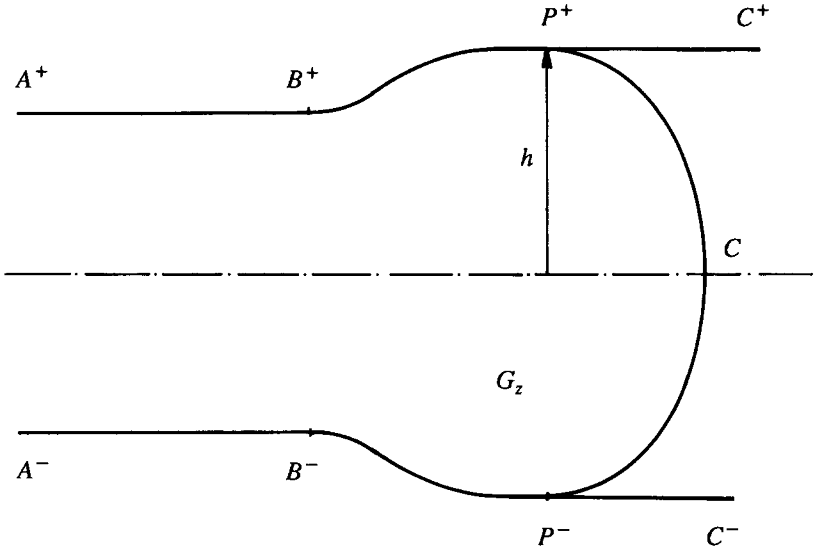

Fig. 3.3. The domain G_z under $m_N(\zeta)$.

not correspond to the parts P^+C^+ and P^-C^- of the exact free boundary. In order to obtain the complete shape of the fluid surface, we can add two straight horizontal lines from P^+ and P^- to the right as shown in Fig. 3.3.

In the approximation theory the angle α, $0<\alpha<\frac{1}{2}\pi$, is determined by the condition that the y-coordinate along the boundary CP^+B^+ attains its maximum in the point P^+; so

$$\frac{\mathrm{d}y}{\mathrm{d}\alpha}=\mathrm{Im}\left[\frac{\mathrm{d}z}{\mathrm{d}\alpha}\right]=\mathrm{Im}[i\,\mathrm{e}^{i\alpha}\mathrm{m}_N'(\mathrm{e}^{i\alpha})]=0\,,\quad 0<\alpha<\tfrac{1}{2}\pi\,. \tag{3.5}$$

The swell-ratio h is then calculated as the maximum value of the y-coordinate,

$$h=\mathrm{Im}[\mathrm{m}_N(\mathrm{e}^{i\alpha})]\,. \tag{3.6}$$

Finally, we remark that the polynomial mapping $z=\mathrm{m}_N(\zeta)$ must be conformal, i.e. analytic and univalent for $\zeta\in G_\zeta^+$. Since $\mathrm{m}_N(\zeta)$ is an analytic function for all $\zeta\in\mathbf{C}$, only the univalence has to be shown. This property is equivalent to the statement that the derivative $\mathrm{m}_N'(\zeta)$ doesn't vanish. This condition will be verified in the final section after the determination of the coefficients μ_k, $0\leq k\leq N$.

4. Solution of the problem

We follow the procedure of solution as employed in [8]. Approximating the exact mapping function $z=\mathrm{m}(\zeta)$ by a polynomial $\mathrm{m}_N(\zeta)$, we replace the functions $\Omega_1(z)$ and $\omega_1(z)$ by $\Omega_N(\zeta)$ resp. $\omega_N(\zeta)$. These two functions are analytic for $\zeta\in G_\zeta^+$, but will not be polynomials in general. The boundary conditions as formulated in section 2 must be transformed properly to the unit circle, $|\xi|=1$. From a combination of (2.17) (i) and (ii) we find

$$w_1=\frac{\mathrm{m}_N^+(\xi)\overline{\Omega_N'^{\,+}(\xi)}+\overline{\omega_N'^{\,+}(\xi)}}{\overline{m_N'^{\,+}(\xi)}}-\Omega_N^+(\xi)=0\,,\quad \xi\in B^+AB^-\,. \tag{4.1}$$

The arcs B^+P^+ and B^-P^- correspond to a part of the fluid surface; so boundary condition (2.9) holds along these arcs. The arc P^-CP^+ does not correspond to the free boundary which stretches out to infinity. Since this arc is situated far from the die opening, we impose condition (2.12) there. Consequently, condition (2.9) also holds on P^-CP^+. Because of the continuity of the normal and shear stresses along the arcs B^-P^-, P^-CP^+, and B^+P^+ of the unit circle, boundary conditions (2.12) and (2.18) transform into

$$K_1 = \frac{\mathrm{m}_N^+(\xi)\overline{\Omega_N'^{+}(\xi)} + \overline{\omega_N'^{+}(\xi)}}{\overline{\mathrm{m}_N'^{+}(\xi)}} + \Omega_N^+(\xi) = g_N(\xi)\,, \quad \xi \in \mathrm{B^-CB^+}\,, \tag{4.2}$$

with $g_N(\xi)$ defined by

$$g_N(\xi) := \tfrac{3}{8}([\mathrm{m}_N^+(\xi)]^2 + 2\mathrm{m}_N^+(\xi)\overline{\mathrm{m}_N^+(\xi)} - [\overline{\mathrm{m}_N^+(\xi)}]^2)\,, \quad \xi \in \mathrm{B^-CB^+}\,. \tag{4.3}$$

The problem stated above can be solved by an analytic continuation of $\Omega_N(\zeta)$ to the exterior of the unit circle G_ζ^-. This continuation is denoted by $\Psi_N(\zeta)$ and is defined by

$$\Psi_N(\zeta) := \begin{cases} \Omega_N(\zeta)\,, & \zeta \in G_\zeta^+\,, \\ \dfrac{\mathrm{m}_N(\zeta)\overline{\Omega_N'(1/\bar\zeta)} + \overline{\omega_N'(1/\bar\zeta)}}{\overline{\mathrm{m}_N'(1/\bar\zeta)}}\,, & \zeta \in G_\zeta^-\,. \end{cases} \tag{4.4}$$

The function $\Psi_N(\zeta)$ is analytic for $\zeta \in G_\zeta^+ \cup G_\zeta^-$ and must satisfy the holomorphy condition, see [4, Sec. 5.4], [8],

$$\omega_N'(\zeta) = \mathrm{m}_N'(\zeta)\overline{\Psi_N(1/\bar\zeta)} - \overline{\mathrm{m}_N(1/\bar\zeta)}\Psi_N'(\zeta) = \mathrm{O}(1)\,, \quad (\zeta \to 0)\,. \tag{4.5}$$

By definition (4.4) the boundary conditions (4.1) and (4.2) become

$$\Psi_N^-(\xi) - \Psi_N^+(\xi) = 0\,, \qquad \xi \in \mathrm{B^+AB^-}\,, \tag{4.6}$$

$$\Psi_N^-(\xi) + \Psi_N^+(\xi) = g_N(\xi)\,, \quad \xi \in \mathrm{B^-CB^+}\,. \tag{4.7}$$

Near the separation points B^+ and B^- the velocity must remain finite. Therefore, we have

$$\Psi_N(\zeta) = \mathrm{O}(1)\,, \quad (\zeta \to \pm i)\,. \tag{4.8}$$

The condition (4.6) implies that the function $\Psi_N(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash \mathrm{B^-CB^+}$. The jump condition (4.7) over the arc B^-CB^+ and the condition (4.8) near the endpoints B^- and B^+ determine a so-called Hilbertproblem for the function $\Psi_N(\zeta)$. For a detailed description of the theory for the solution of Hilbert problems we refer to Muskhelishvili [5, Ch. 18] and England [4, Ch. 1]. The solution of this Hilbert problem is derived in [8] and can be represented by

$$\Psi_N(\zeta) = X(\zeta)G_N(\zeta) + X(\zeta)F_N(\zeta)\,, \quad \zeta \in \mathbb{C}\backslash \mathrm{B^-CB^+}\,, \tag{4.9}$$

where $G_N(\zeta)$ is defined by

$$G_N(\zeta) = \frac{1}{2\pi i} \int_{B^-CB^+} \frac{g_N(\xi)}{X^+(\xi)(\xi - \zeta)} \, d\xi \,, \quad \zeta \in \mathbf{C} \backslash B^-CB^+ \,. \tag{4.10}$$

Evaluating the integral (4.10) by means of contour integration, we can express the function $G_N(\zeta)$ in the coefficients μ_k, $0 \leq k \leq N$, of the conformal mapping. The function $X(\zeta)$ is the characteristic Plemelj function defined by

$$X(\zeta) = (\zeta - i)^{1/2}(\zeta + i)^{1/2} \,, \quad \zeta \in \mathbf{C} \backslash B^-CB^+ \,, \tag{4.11}$$

and has a branch cut along the arc B^-CB^+. The function $F_N(\zeta)$ is a polynomial of degree $N-1$,

$$F_N(\zeta) = \sum_{k=0}^{N-1} f_k \zeta^k \,, \quad \zeta \in \mathbf{C} \,. \tag{4.12}$$

The coefficients of this function are determined by the holomorphy condition (4.5). This condition yields N linear equations for the unknown coefficients f_k, $0 \leq k \leq N-1$. Solving these equations we find an explicit formula for the function $\Psi_N(\zeta)$ in terms of the coefficients μ_k, $0 \leq k \leq N$, from the relations (4.9) and (4.10).

In order to determine the coefficients μ_k, $0 \leq k \leq N$, of the conformal mapping function $\mathrm{m}_N(\zeta)$ and thereby the shape of the fluid surface, we need $N+1$ algebraic equations. Since the point $\zeta = i$ corresponds to the separation point B^+, $z = i$, we have

$$\mathrm{m}_N(i) = i \,. \tag{4.13}$$

The point $\zeta = -i$, is then mapped onto B^-, $z = -i$. Two equations are supplied by the real and imaginary parts of (4.13), so $N-1$ more equations are required. The boundary B^+P^+ forms a free streamline, whose shape is determined by condition (2.11). Therefore, we demand that the normal velocity vanishes in $N-1$ points of the boundary B^+P^+. This condition is then also satisfied on the arc B^-P^- as a consequence of the symmetry of the problem. We choose the following points

$$\xi_k := e^{i\theta_k} \,, \quad \theta_k := \alpha + \left(\frac{1}{2}\pi - \alpha\right)\frac{k}{N} \,, \quad 1 \leq k \leq N-1 \,, \tag{4.14}$$

where α is the angle introduced in Section 3.

The free-streamline condition (2.11) is transformed to the ζ-plane with relation (3.2). The coefficients μ_k, $0 \leq k \leq N$, are now determined by (4.13) and by the conditions

$$\mathrm{Re}[w_N \overline{\xi_k \mathrm{m}_N'^{+}(\xi_k)}] = 0 \,, \quad 1 \leq k \leq N-1 \,, \tag{4.15}$$

where w_N is the complex velocity

$$w_N = \tfrac{3}{2} + \tfrac{3}{8}([\mathrm{m}_N^+(\xi)]^2 - 2\mathrm{m}_N^+(\xi)\overline{\mathrm{m}_N^+(\xi)} + [\overline{\mathrm{m}_N^+(\xi)}]^2) + \Psi_N^-(\xi) - \Psi_N^+(\xi) \,. \tag{4.16}$$

These equations are solved by a numerical procedure for the solution of systems of non-linear equations. An approximation of the shape of the free boundary $B^+P^+(B^-P^-)$ is then given by the relation (3.4). The results are presented in the final section.

5. Results and conclusions

In this section we present the results of the polynomial approximation of the conformal mapping function. The coefficients μ_k, $0 \leqslant k \leqslant N$, of the function $\mathrm{m}_N(\zeta)$ and the angle α defined by equation (3.5) are calculated for $N=6$, 7, and 8 and are listed in Table 5.1. Estimates of the errors in the values of the coefficients μ_k are in the order of 10^{-5}, if $N=6$, and in the order of 10^{-4}, if $N=7$ or 8. This means that the error in μ_k is 0.1% or less. Taking $N \leqslant 5$, we did not obtain any trustworthy outcome, because the degree of approximation was apparently too low. On the other hand, no improvement was observed in the case $N=9$ or 10. So we conclude that the approximations for $N=6$, 7, 8 produce reliable results with rather little calculus.

The mapping function $z=\mathrm{m}_N(\zeta)$ has to be conformal, see Section 3. Therefore, it must be examined, if zeroes of the derivative $\mathrm{m}_N'(\zeta)$ occur in the domain G_ζ^+. The total number of zeroes of $\mathrm{m}_N'(\zeta)$ inside the unit circle Γ is given by the integral

$$I_1 = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{m}_N''(\zeta)}{\mathrm{m}_N'(\zeta)}\,\mathrm{d}\zeta\,. \tag{5.1}$$

Calculation of the integral I_1 yields that there is one single zero in G_ζ^+ for $N=6$, 7 as well as $N=8$. The position of the zero $\zeta=\zeta_0$ is given by the integral

$$I_2 = \frac{1}{2\pi i}\int_\Gamma \frac{\zeta\mathrm{m}_N''(\zeta)}{\mathrm{m}_N'(\zeta)}\,\mathrm{d}\zeta\,. \tag{5.2}$$

We find $\zeta_0=-0.04$, $\zeta_0=+0.03$, $\zeta_0=+0.02$ for $N=6$, 7, 8 respectively. These points are all situated near the origin $\zeta=0$ and so they are mapped onto points in the neighbourhood of $z=\mathrm{m}_N(0)=\mu_0$ in the domain G_z. Since the points $z=\mu_0$ have great distance to the separation points B^+ and B^-, we conclude that the mapping function $z=\mathrm{m}_N(\zeta)$ is conformal near the arcs $\mathrm{B}^+\mathrm{P}^+$ and $\mathrm{B}^-\mathrm{P}^-$.

The shape of the free boundaries $\mathrm{B}^+\mathrm{P}^+$ and $\mathrm{B}^-\mathrm{P}^-$ is calculated with relation (3.4) and is shown in Fig. 5.1. The difference between the approximations of the free boundary is about 3–4%. The approximation h for the swell-ratio follows from equation (3.6). Another estimate H for the swell-ratio is based on the property (2.13) that the velocity at infinity is

Table 5.1. The coefficients μ_k and the angle α for several values of N

k	$N=6$	$N=7$	$N=8$
0	1.0088	1.2603	1.1586
1	0.1509	−0.1395	−0.0760
2	1.7929	2.2748	2.2280
3	−1.1489	−1.6110	−1.6411
4	0.8997	1.2085	1.3723
5	−0.2998	−0.5032	−0.6470
6	0.1157	0.1941	0.3261
7		−0.0307	−0.0812
8			0.0234
α/π	0.2514	0.2761	0.2540

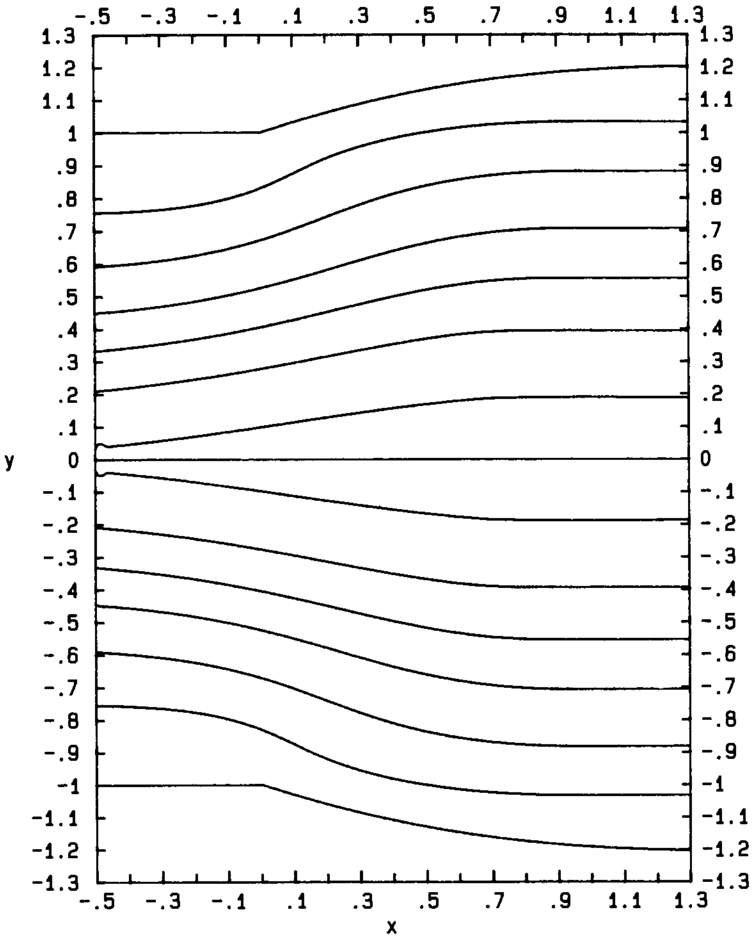

Fig. 5.1. The free boundary.

uniform. This estimate H is then defined as the reciprocal value of the velocity in the point C, $\zeta = +1$. The values of h and H are listed in Table 5.2.

The results are compared with swell-ratios tabulated by Tanner [7, sec. 8.3], who gives finite-element simulations obtained by the following authors. Crochet and Keunings found a swell-ratio of 1.188 in [2] and several values between 1.196 and 1.227 in [3]. Chang, Patten and Finlayson [1] calculated a swell-ratio of 1.206, while Reddy and Tanner [6] obtained 1.199. So we conclude, that the exact swell-ratio for an incompressible Newtonian fluid will be 1.20 with an error of 2% at most. This is in full agreement with the numerical results which lie in the same range.

Table 5.2. Estimates for the swell-ratio

N	h	H
6	1.1835	1.1982
7	1.2245	1.2247
8	1.2026	1.2027

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